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Abstract

We consider scalar quantum fields with exponential interaction on Euclidean hyperbolic space $\mathbb{H}^2$ in two dimensions. Using decoupling inequalities for Neumann boundary conditions on a tesselation of $\mathbb{H}^2$, we are able to show that the infra-red limit for the generating functional of the conformal boundary field becomes trivial.

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1 Introduction

One motivation for the study of the AdS/CFD correspondence originally proposed by J. Maldacena in the context of string theory [15] in the framework of Euclidean, constructive quantum field theory [7] is the hope to discover new, interacting and at the same time confomally invariant boundary theories. In this article we show that this program is subject to a new class of infra red divergences leading to trivial generating functionals at the conformal boundary. This was already noted in [8], however the proof given in this reference for ϕ^4-theory requires an ultra violet cut off for technical reasons. In this article we for the first time derive a related triviality result for the exponential interaction with sufficiently small coupling on the two dimensional hyperbolic space without any cut offs.

In a previous work [8], following the outline given in [5], we proved that the following functional integral describes the AdS/CFT-correspondence for scalar fields [5, 12, 19] both from a "scaling to the conformal boundary" and a "prescription of boundary values" point of view

$$\begin{aligned}\tilde{Z}(h,V_\Lambda)/\tilde{Z}(0,V_\Lambda) &= \lim_{z\to 0} e^{-\mathrm{Corr}(h,h)} \int_{\mathscr{D}'} e^{-V_\Lambda(\phi)} e^{\phi(z^{-\Delta_+}\delta_z\otimes h)} d\mu_+(\phi)/\tilde{Z}(0,V_\Lambda) \\ &= e^{\frac{1}{2}\alpha_+(h,h)} \int_{\mathscr{D}'} e^{-V_\Lambda(\phi+H_+h)} d\mu_+(\phi)/\tilde{Z}(0,V_\Lambda). \end{aligned} \tag{1}$$

Here, $\Delta_+ = \frac{d-1}{2} + \frac{1}{2}\sqrt{(d-1)^2+4m^2}$ is a conformal weight, V_Λ is an interaction restricted to a bounded region Λ, and $\mathscr{D}' = C_0^\infty(\mathbb{H}^d)'$ stands for the space of non-tempered distributions over the d dimensional hyperboloc space $\mathbb{H}^d$, cf Appendix A. In the following we restict to the exponential interaction [2] and $d = 2$ [1]. μ_+ is the Gaussian measure on $\mathscr{D}'$ with covariance operator $G_+ = (-\Delta_{\mathbb{H}^2} + m^2)^{-1}$ with boundary conditions fo $\Delta_{\mathbb{H}^2}$ fixed by (10) and (11) below. H_+ is the bulk-to-boundary propagator which accounts for the way how fluctuations in the bulk are transferred to the boundary and α_+ is the boundary-to-boundary propagator, [5, 8]. $\mathrm{Corr}(h,h)$ is some z-dependent correction factor and thus does not change the relativistic field content. It is however a necessary regularization factor for the Euclidean theory, even in the case of non interacting fields. The variable z is taken from the half-space model of $\mathbb{H}^2$, cf. Appendix A. The reason why (1) is qualified as the generating functional of a field theory with conformal invariance properties on the boundary $\partial_c\mathbb{H}^2$ rests essentially on the following two properties:

- Functional (1) is reflection positive (not necessarily stochastically positive).
- It obeys conformal invariance on $\partial_c\mathbb{H}^2$ in the following sense

$$\tilde{Z}(h,V_\Lambda)/\tilde{Z}(0,V_\Lambda) = \tilde{Z}(\lambda_u^{-1}uh, V_{u\Lambda})/\tilde{Z}(0,V_{u\Lambda}), \tag{2}$$

  where λ_u is a conformal density depending on $u \in O^+(2,1) = \mathrm{Iso}(\mathbb{H}^2)$.

In fact, if the following limit exists uniquely w.r.t. to nets $\Lambda \uparrow \mathbb{H}^2$, of bounded measurable subsets,

$$\tilde{Z}_{\lim}(h) = \lim_{\Lambda\to\infty} \tilde{Z}(h,V_\Lambda)/\tilde{Z}(0,V_\Lambda), \tag{3}$$

then property (2) entails that the limit functional satisfies reflection positivity and conformal invariance with respect to the induced conformal group action of $O^+(2,1)$ on the boundary, cf. [8, 9]. Still, this infra-red limit $\tilde{Z}_{\lim}$ may turn out to be trivial, revealing that the AdS/CFT-prescription is not meaningful, at least for the construction of conformal fields from fields that are defined on fixed $\mathbb{H}^2$-backgrounds. In [9] we obtained a partial result in this direction when the UV-regularized potential $V_\Lambda =: \phi^4 :$ is considered. Namely, in this case

$$\tilde{Z}_{\lim}(h) = \begin{cases} 0 & \text{for } h \neq 0; \\ 1 & \text{for } h = 0. \end{cases} \tag{4}$$

As will be shown in this article this turns out to be true also for exponential interactions without cut-offs at small coupling.

The paper is organized as follows: In Section 2 we define Euclidean functional integrals with free and Neumann boundary conditions on a tesselation of $\mathbb{H}^2$. In Section 3 we construct the exponential interaction on $\mathbb{H}^2$ and apply decoupling inequalities. In Section 4 we derive the triviality theorem for the generating functional $Z_{\text{lim}}(h)$ under the net limit $\Lambda \uparrow \mathbb{H}^2$ in the case of small coupling, which is the main result of this article.

2 Tessellations and the Neumann Green's Function

Since the proof of Theorem 4.1 below strongly relies on a decoupling of Neumann fields along isometric regions, we first provide some geometric features regarding regular tessellations. Here a tessellation of $\mathbb{H}^2$ is a family $(T_j)_{j\in\mathbb{N}}$ of convex polygons obeying

$$\mathbb{H}^2 = \bigcup_{j\in\mathbb{N}} T_j, \quad \mathring{T}_i \cap \mathring{T}_j = \emptyset \quad \text{for } i \neq j.$$

Regular means that the T_j's are congruent, i.e., for all $j, k \in \mathbb{N}$ there is an isometry $g \in \mathrm{SO}(2,1)$ with $g(T_j) = T_k$. In this case $\mathring{T}_1$ is called a fundamental domain. The polygons are formed by n vertices together with n sides which are simply geodesic segments. Suppose we consider the angle between the two geodesics that pass through a given vertex and are perpendicular to the sides that have this vertex in common. If all these angles are of the form $\pi/k, k \in \mathbb{N}$, then a tessellation can be generated from the compact polygon T_1 by repeated reflections in its sides, see [17, Theorem 7.1.3]. Note that these reflections are isometries. First one reflects in the sides of T_1, then in the sides of the new T_j's that have just been generated and so on. By gathering all possible compositions of reflections into a group we obtain the reflection group Γ related to the tessellation. An example of a tessellation by means of hyperbolic triangles is given in Figure 1.

In the following we assume that the tessellation and corresponding reflection group Γ on $\mathbb{H}^2$ are given by means of a compact polygon as described above. We are ready to define a Green's function G_N that satisfies the Neumann boundary conditions on $\bigcup_{j\in\mathbb{N}} \partial T_j$. For this we first define

$$G_{N,j}(x,y) := \begin{cases} \sum_{\gamma\in\Gamma} G_+(x,\gamma(y)), & \text{if } x \neq y \in T_j \\ +\infty, & \text{if } x = y \in T_j \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Then, for $x, y \in \mathbb{H}^2$, we set

$$G_N(x,y) := \sum_{j\in\mathbb{N}} G_{N,j}(x,y). \tag{6}$$

Let $N(\vartheta, x, y) := \operatorname{card}\{\gamma \in \Gamma |\, \rho(x,\gamma(y)) < \vartheta\}$ be the orbital counting function. For $m^2 > 0$ convergence of the sum in (5) can be seen by combining the following bound, cf. [16, Theorem 1.5.1],

$$N(\vartheta, x, y) < Ae^{\vartheta}, \quad A > 0, \tag{7}$$

with the fact that $G_+(x,y) \sim \text{const.}\, e^{-\Delta_+\rho(x,y)}$ for large geodesic distances $\rho(x,y)$, see Appendix A.

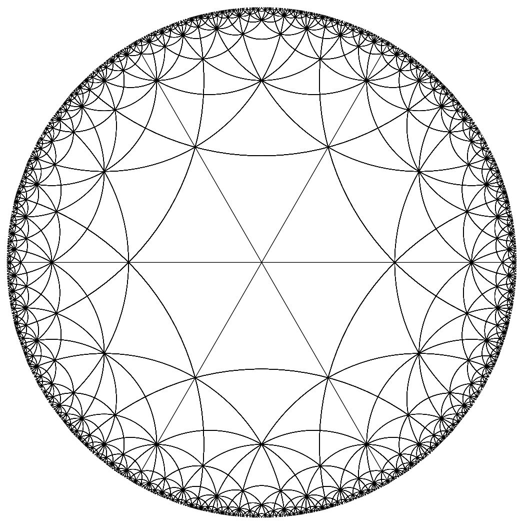

Figure 1: A tessellation constructed by reflections of triangles with angles $\pi/3, \pi/4, \pi/4$.

Next we need to check the basic properties of a Neumann Green's function. The invariance property $G_+(x,y) = G_+(u(x), u(y))$, for $u \in \mathrm{Iso}(\mathbb{H}^2)$, immediately entails the symmetry of G_N. Given $x, y \in \mathring{T}_j$, then each $T_k, k \neq j$, contains precisely one of the reflected points so that

$$(-\Delta_{\mathbb{H}^2} + m^2) G_N(x,y) = (-\Delta_{\mathbb{H}^2} + m^2) G_+(x,y) = \delta(x,y). \tag{8}$$

Since G_+ has a logarithmic singularity, see the Appendix, we find for the same reason that $G_N(x,y) \sim -1/(2\pi)\log(\rho(x,y))$, as $\rho(x,y) \to 0$.

In order to see that (6) satisfies Neumann boundary conditions we consider any normal derivative w.r.t. an arbitrary side s. For this we take any geodesic $y \equiv y(t)_{-t_0 \leq t \leq t_0}$ with $t_0 > 0$ such that y intersects s perpendicularly at $t = 0$. Then, if $\tilde{\gamma}$ denotes reflection in the side s we have $\tilde{\gamma}(y(-t)) = y(t)$. Let us define the function

$$f(t) := \begin{cases} G_N(x, y(t)), & \text{if } t \leq 0, \\ G_N(\tilde{\gamma}(x), y(t)), & \text{if } t > 0. \end{cases} \tag{9}$$

Now, owing to the invariance $G_N(x,y) = G_N(\tilde{\gamma}(x), \tilde{\gamma}(y))$ it follows that f is an even function w.r.t. $t = 0$, so that its derivative has to vanish at this point, which is what we wanted to verify.

As can be seen from uniqueness of the Neumann problem, $G_N(x,y)$ is the integral kernel of $(-\Delta_N + m^2)^{-1}$, where $-\Delta_N$ is the Laplacian with Neumann boundary conditions on $\bigcup_{j\in\mathbb{N}} \partial T_j$. The operators $-\Delta_{\mathbb{H}^2}$ and $-\Delta_N$ are associated with the following quadratic forms

$$\mathscr{B}_+(f,g) = \mathscr{B}_N(f,g) = \int_{\mathbb{H}^2} \langle \nabla f, \nabla g \rangle \, dx, \tag{10}$$

with $\langle .,. \rangle$ the canonical scalar product on $T\mathbb{H}^2$ and form domains given by

$$\mathscr{D}_+ = H^1(\mathbb{H}^2) \subset \bigoplus_{j\in\mathbb{N}} H^1(T_j) = \mathscr{D}_N, \tag{11}$$

where we have introduced the Sobolev space $H^1(\mathbb{H}^2) = \{f \in L^2(\mathbb{H}^2) |\, \nabla f \in L^2(\mathscr{X}(\mathbb{H}^2))\}$, with $L^2(\mathscr{X}(E))$ denoting the space of square intagrable vector fields on $E \subset \mathbb{H}^2$. Moreover, $H^1(T_j)$ consists of those $f \in L^2(T_j)$ with weak derivative $\nabla f \in L^2(\mathscr{X}(T_j))$. The embedding (11) is realized through the mapping $f \mapsto \oplus_{j\in\mathbb{N}} f|_{T_j}$. Let us recall the following comparison theorem, see [14, Ch.6, Theorem 2.21].

Theorem 2.1 *Let $\mathscr{B}_A$ and $\mathscr{B}_B$ be two quadratic forms defined on a Hilbert space H with form domains $\mathscr{D}_A$ and $\mathscr{D}_B$, respectively. If $\mathscr{D}_A \subset \mathscr{D}_B$ and $\mathscr{B}_A(f,f) \geq \mathscr{B}_B(f,f) \geq \alpha$ for $\alpha \in \mathbb{R}$ and all $f \in \mathscr{D}_A$, then*

$$(A+\zeta)^{-1} \leq (B+\zeta)^{-1}, \quad \forall\, \zeta < \alpha,$$

where A, B are the operators associated with the forms $\mathscr{B}_A$ and $\mathscr{B}_B$, respectively.

The L^2 spectrum of $-\Delta_{\mathbb{H}^2}$ is $[1/4, \infty)$, cf. [4, Theorem 5.7.1]. Therefore, Theorem 2.1, applied with $A = (-\Delta_{\mathbb{H}^2} + m^2)$ and $B = (-\Delta_N + m^2)$, shows that

$$G_+ = (-\Delta_{\mathbb{H}^2} + m^2)^{-1} \leq (-\Delta_N + m^2)^{-1} = G_N, \quad \text{for } m^2 > -1/4. \tag{12}$$

Inequality (12) allows to apply the theory of conditioning as described in [18] or in [10]. According to the latter we can write $\phi_N(f) = \phi_+(f) + \phi_R(f)$, where $R = G_N - G_+$ and the random fields are indexed by a common Hilbert space $\widetilde{H}$. The precise definitions are as follows. Let H_N, H_+ and H_R be the Hilbert spaces that are obtained upon completing $C_0^\infty(\mathbb{H}^2)$ w.r.t. the norms $\|f\|_N = G_N(f,f)^{\frac{1}{2}}, \|f\|_+ := G_+(f,f)^{\frac{1}{2}}$ and $\|f\|_R := G_R(f,f)^{\frac{1}{2}}$, respectively. Let $\widetilde{H} := H_+ \oplus H_R$ equipped with the direct sum norm, denoted by $\|\cdot\|_{\widetilde{H}}$. These Hilbert spaces are accompanied by measure spaces $(Q_\natural, \mathscr{Q}_\natural, \mu_\natural)$, on which the random fields $\phi_\natural$ are defined as random variables. The symbol $\natural$ indicates one of the Hilbert spaces, such that the $\mu_\natural$'s are the measures associated with $G_\natural$. For $\natural = +, N, R$ we consider $(Q_\natural, \mathscr{S}_\natural) = (\mathscr{D}', \mathscr{B})$, where $\mathscr{B}$ is the Borel σ-algebra generated by the weak*-topology of $\mathscr{D}'$. Especially, $\mu_{\widetilde{H}} = \mu_+ \otimes \mu_R$, where the latter is defined on $(Q_+ \times Q_R, \mathscr{Q}_+ \otimes \mathscr{Q}_R)$. Since it holds that $G_+ \leq G_N$ and $G_R \leq G_N$, each $f \in H_N$ can be identified with unique elements $f_+ \in H_+$ and $f_R \in H_R$. In other words there is a natural embedding $H_N \hookrightarrow \widetilde{H}$ given by $f \mapsto (f_+, f_R)$ so that the Neumann field should correctly be written as $\phi_N(f) := \phi_{\widetilde{H}}(f_+, f_R) = \phi_+(f_+) + \phi_R(f_R)$. If $P_+ f := (f_+, 0)$, the projection on the first component, then obviously $\phi_N(P_+ f) = \phi_+(f_+)$. Therefore one says that ϕ_+ is obtained from ϕ_N by conditioning. Even more is true as will be explicated in the next section. In the sequel we shall simply write $\mu = \mu_{\widetilde{H}}, Q = Q_+ \times Q_R, \mathscr{Q} = \mathscr{Q}_+ \otimes \mathscr{Q}_R, \phi = \phi_{\widetilde{H}}$.

3 The exponential interaction and a conditioning estimate

Below $\phi_\natural$ will denote one of the fields ϕ_+ or ϕ_N. In order to define the exponential interaction we start from the kth Wick power $:\phi_\natural^k:(g)$. Here it is tacitly understood that the Wick ordering is taken with respect to the Green function $G_\natural$. In the previous section we recalled that ϕ_N can also be realized as a random variable on the measure space $(Q, \mathscr{Q}, \mu)$. Therefore, without any further notice, statements regarding $L^2(\mu_N)$-limits will at the same time be regarded as statements about $L^2(\mu)$-limits. As the following lemma shows the exponential interaction can be defined in terms of the series

$$:\exp(\alpha\phi_\natural):(g) := \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} :\phi_\natural^k:(g). \tag{13}$$

Lemma 3.1 *Assume that $\Lambda \subset \mathbb{H}^2$ is a compact measurable set and let $g \in L^{1+\varepsilon}(\mathbb{H}^2, dx)$, where $\varepsilon > 0$. For $d = 2, |\alpha| < \sqrt{4\pi}$ the following statements hold*

(i) The Wick power $:\phi_\natural^k(g):$ exists in $L^p(\mu_\natural)$ for any $k \in \mathbb{N}_0$ and $0 \le p < \infty$.

(ii) $:\exp(\alpha\phi_\natural):(g)$ exists in $L^2(\mu_\natural)$. In particular

$$:\exp(\alpha\phi_\natural):(1_\Lambda g) \equiv \int_\Lambda :\exp(\alpha\phi_\natural(x)):g(x)dx \tag{14}$$

is a well defined $L^2(\mu_\natural)$ random variable.

(iii) $\int_\Lambda :\exp(\alpha\phi_{\natural,\varepsilon}(x)):g(x)dx = \int_\Lambda \frac{\exp(\alpha\phi_{\natural,\varepsilon}(x))g(x)}{\exp(\frac{\alpha^2}{2}G_{\varepsilon,\natural}(x,x))}dx \to \int_\Lambda :\exp(\alpha\phi_\natural(x)):g(x)dx$, *as $\varepsilon \to 0$ in $L^2(\mu_\natural)$.*

Remark: The smoothed fields $\phi_{\natural,\varepsilon}$ are defined as $\phi_{\natural,\varepsilon} = \chi_\varepsilon * \phi_\natural$, where $(\chi_\varepsilon)_{\varepsilon>0}$ is a family of nonnegative functions from $C_0^\infty(\mathbb{H}^2)$, which approximate δ_o the Dirac distribution at the origin o. Further we shall assume that the integral of each member χ_ε is one, since in this case the norm of the operator $\mathscr{T}_\varepsilon(f) := \chi_\varepsilon * f$ is bounded by one in any $L^p(\mathbb{H}^d, dx) \equiv L^p$ space with $p \in [1, \infty]$, see the statement after inequality (33).

Proof of Lemma 3.1. (i) The kth Wick power $:\phi_\natural^k:(g)$ is defined as the unique element in $\mathcal{H}_k^\natural = H_\natural^{\otimes k}$ such that

$$\langle :\phi_\natural^k:(g), :\phi_\natural(h_1)\cdots\phi_\natural(h_k):\rangle = k! \int_{(\mathbb{H}^2)^{k+1}} g(x) \prod_{j=1}^{k} G_\natural(x, y_j)h_j(y_j)dy_j dx, \quad \text{for all } h_j \in \mathscr{D}. \tag{15}$$

A sufficient condition for $:\phi_\natural^k:(g)$ to exist is given by the ensuing bound, cf. [18, Proposition V.1]

$$\int_{(\mathbb{H}^2)^2} g(x)G_\natural(x,y)^k g(y)dydx \le \text{const.}|||g|||, \tag{16}$$

with $|||\cdot|||$ denoting a norm that is continuous on $\mathscr{D}$. If the latter bound is valid then, as will be shown below, the $L^2(\mu_\natural)$-norm can be calculated by

$$\| :\phi_\natural^k: (g)\|_{L^2(\mu_\natural)} = k! \int_{(\mathbb{H}^2)^2} g(x) G_\natural(x,y)^k g(y) dy dx. \tag{17}$$

Since $G_N \le cG_+$[1], for some constant $c > 0$, we may reduce the proof of existence of $:\phi_N^k:$, by a conditioning argument, to that of $:\phi_+^k:$. In fact, by the conditioning comparison result [10, Theorem III.1] one gets $\| :\phi_N^k(g): \|_{L^p(\mu_N)} \le \| :\phi_+^k(g): \|_{L^p(\mu_{c+})}$, where μ_{c+} is the measure related to cG_+. By hypercontractivity it is possible to estimate $\| :\phi_+^k(g): \|_{L^p(\mu_{c+})}$ in terms of $\| :\phi_+^k(g): \|_{L^2(\mu_+)}$, see the proof of Lemma III.7 in [10], so that we only need to show existence for the ϕ_+ field. Due to left-invariance of $G_+(x,y)$ we may always shift y to a fixed origin $o \in \mathbb{H}^2$, so that G_+ becomes a function of one variable. Using convolution on $\mathbb{H}^2$, as described in the Appendix, the integral of (16) can be written as

$$\int_{\mathbb{H}^2} g(x)(g * G_+^k)(x) dx, \quad g \in L^{1+\varepsilon}. \tag{18}$$

Employing Hölder's and Young's inequalities we obtain

$$\int_{\mathbb{H}^2} g(x)(g * G_+^k)(x) dx \le \|g\|_{1+\varepsilon}^2 \|G_+^k\|_q, \quad q = \tfrac{1+\varepsilon}{2\varepsilon}, \tag{19}$$

see [10, Lemma III.7]. Existence of $\|G_+^k\|_q$ can be deduced from the logarithmic singularity and the exponential decay $\sim e^{-\Delta_+\rho}$ of G_+ in combination with the representation $dx = \sinh\rho d\rho d\omega$, where $d\omega$ is the standard measure on $\mathbb{S}^1$. It should be noted that for $g \in \mathscr{D}$ identity (17) is valid, see Proposition 8.3.1 and its Corollaries in [7]. Now, let $g \in L^{1+\varepsilon}$ and let $(g_n)_{n\in\mathbb{N}}, g_n \in \mathscr{D}$, be a sequence with $\lim_{n\to\infty} g_n = g$ in $L^{1+\varepsilon}$. Employing linearity of $:\phi_+^k:(\cdot)$ and the bound (19) we obtain that $:\phi_+^k:(g_n)$ is a Cauchy sequence in $L^p(\mu_+)$. Hence, the limit denoted by $:\phi_+:(g)$ exists. The bilinear form corresponding to the integral of (19) can be bounded by $\|f\|_{1+\varepsilon}\|g\|_{1+\varepsilon}\|G_+^k\|_q$, $f, g \in L^{1+\varepsilon}$. Hence it is continuous and from this it is readily seen that (17) is also valid for $g \in L^{1+\varepsilon}$.
(ii) and (iii) Both cases can be treated along the same lines as in [2]. Assertion (i) follows from equality (17) that leads to

$$\| :\exp(\alpha\phi_\natural): (g)\|_{L^2(\mu)}^2 = \sum_{k=0}^{\infty} \frac{\alpha^{2k}}{k!} (g, G_\natural^k g)_{L^2} = (g, \exp(\alpha^2 G_\natural) g)_{L^2}. \tag{20}$$

The last inner product exists due to the logarithmic singularity of $G_\natural$ for $|\alpha| < \sqrt{4\pi}$. Claim (ii) can be verified following the reasoning in [2, eqs (5.7)-(5.12)]. □

[1] this fact has been proved in the references and carries over to our case, cf. [10, Theorem III.4] and [11, Lemma III.5B]

Lemma 3.2 *If ϕ_+ is obtained from ϕ_N by conditioning then we have for $h \in C^\infty(\partial_c \mathbb{H}^2)$*

$$\int_{Q_+} e^{-V_\Lambda(\phi_+ + H_+ h)} d\mu_+(\phi_+) \le \int_Q e^{-V_\Lambda(\phi_N + H_+ h)} d\mu(\phi). \tag{21}$$

Proof. Lemma 3.1(iii) together with a limiting argument entail that it is sufficient to prove this statement for the smoothed fields $\phi_{+,\varepsilon}$ and $\phi_{N,\varepsilon}$. But for this case the assertion can be proved like in the Appendix of [1]. □

4 Triviality for small coupling

In this section we show that if V_Λ is the exponential interaction with coupling constant $\lambda > 0$ as defined in Lemma 3.1,

$$V_\Lambda(\phi) = \lambda :\exp(\alpha\phi):(1_\Lambda) = \lambda :\exp(\alpha\phi):_+(1_\Lambda), \quad |\alpha| < \sqrt{4\pi}. \tag{22}$$

Then, in the limit when $\Lambda \uparrow \mathbb{H}^2$, the functional (1) tends to zero. In this discussion the finite prefactor $e^{\alpha_+(h,h)}$ in (1) is irrelevant.

Proposition 4.1 *Let $X_j = V_{T_j}/\lambda \in [0,\infty]$ with V_{T_j} being defined as a function of ϕ_N, however with + Wick ordering, i.e.*

$$X_j = :\exp(\alpha\phi_N):_+(1_{T_j}) = :\exp(\alpha\phi_N):(1_{T_j} e^{\frac{\alpha^2}{2}\Delta G_N}). \tag{23}$$

Here $\Delta G(x) = G_N(x,x) - G_+(x,x) \ge 0$. With $k_j := \min_{x\in T_j}(H_+ h)$ and $\Lambda = \bigcup_{j=1}^n T_j$, $\mathscr{L}_{X_1}(s) = \mathbb{E}_\mu[e^{-sX_1}]$ we have

$$0 \le \tilde{Z}(h,\Lambda) = \int_{Q_+} e^{-V_\Lambda(\phi_+ + H_+ h)} d\mu_+(\phi_+) \le \prod_{j=1}^n \mathscr{L}_{X_1}(\lambda k_j). \tag{24}$$

Proof. First we notice that the X_j's are i.i.d. random variables under the measure μ, since the T_j's are congruent and G_N is given by (6). Then, employing Lemma 3.2 and independence, we deduce

$$\begin{aligned} 0 &< \tilde{Z}(h,\Lambda) \le Z_N(h,\Lambda) = \int_Q \prod_{j=1}^n e^{-V_{T_j}(\phi_N + H_+ h)} d\mu(\phi) \\ &= \prod_{j=1}^n \int_Q e^{-V_{T_j}(\phi_N + H_+ h)} d\mu(\phi) \le \prod_{j=1}^n \mathscr{L}_{X_1}(\lambda k_j). \end{aligned} \tag{25}$$

□

Proposition 4.2 *For Λ as above we get for the effective action*

$$-\infty < \log\left(\tilde{Z}(h,\Lambda)\right) - \log\left(\tilde{Z}(0,\Lambda)\right) \le \sum_{j=1}^n \left[\log(\mathscr{L}_{X_1}(\lambda k_j)) + \lambda |T_1|\right]. \tag{26}$$

Proof. Just employ (24) and Jensen's inequality

$$\tilde{Z}(0,\Lambda) = \mathbb{E}_{\mu_+}\left[e^{-V_\Lambda}\right] \geq \exp\left\{-\mathbb{E}_{\mu_+}\left[V_\Lambda\right]\right\} = e^{-\lambda|\Lambda|}.$$

□

Theorem 4.1 *("Triviality") Let the coupling constant λ fulfill*

$$0 < \lambda < \frac{-\log(\mu(X_1 = 0))}{|T_1|}, \tag{27}$$

where $\varepsilon > 0$ can be chosen arbitrarily small. Let $h \in C^\infty(\partial_c \mathbb{H}^2)$ with $h > 0$ on a non-degenerate segment (α_0, α_1) of $\partial_c \mathbb{H}^2 \simeq \mathbb{S}^1$. Then there exists a sequence of sets $\Lambda_q \uparrow \mathbb{H}^2$ such that

$$\lim_{q\to\infty} \tilde{Z}(h,\Lambda_q)/\tilde{Z}(0,\Lambda_q) = 0. \tag{28}$$

Remark. The interval (α_0, α_1) stands for the open subset of $\mathbb{S}^1$ whose points have angle between α_0 and α_1. For the proof below we shall work in the disk (ball) model, i.e., $\mathbb{H}^2 = \{x \in \mathbb{R}^2 | \, \|x\| < 1\} =: \mathbb{B}^2$ with boundary $\partial_c \mathbb{H}^2 = \mathbb{S}^1$, see the Appendix. We need to introduce the notion of "conical limit points". Suppose $B(x,\delta)$ denotes a hyperbolic ball of radius δ and center x. The point $p \in \mathbb{S}^1$ is called a conical limit point for Γ if there is an $a \in \mathbb{B}^2$, a sequence $(\gamma_i)_{i\in\mathbb{N}}$ of elements of Γ, a geodesic σ in $\mathbb{B}^2$ ending at p, and a constant $c > 0$ such that $(\gamma_i(a))_{i\in\mathbb{N}}$ converges to p within the c-neighborhood $N(\sigma, c) = \left\{\bigcup_{b\in\sigma} B(b,\delta) | \, \delta < c\right\}$ of σ in $\mathbb{B}^2$. In fact, in this case it can be shown that for each geodesic μ ending at p, there is a constant $t > 0$ such that $(\gamma_i(o))_{i\in\mathbb{N}}$ converges within $N(\mu, t)$. Hence we may assume, without loss of generality, that σ is the segment of the line containing o and p. For the reflections groups we are considering it further holds that "the set of conical limit points" $= \mathbb{S}^1$, cf. [16, Theorem 2.4.8].

Proof of Theorem 4.1. Step 1. By the preceding remark we can find, for an arbitrary point $p \in \mathbb{S}^1$, a sequence $(\gamma_i(a))_{i\in\mathbb{N}}$ that converges to p in the sense described above. If necessary, we rotate our disk such that $p \in (\alpha_0, \alpha_1)$, while keeping the position of h fixed. Let us consider the sector, denoted by $S(r_0)$, which in Euclidean polar coordinates (r, α) is given by $S(r_0) = \{x \in \mathbb{B}^2 | \, r(x) \geq r_0 > 0, \, \alpha(x) \in (\alpha_0, \alpha_1)\}$. Note that the boundary segment at infinity of $S(r_0)$ is naturally identified with (α_0, α_1). We choose one of the polygons, indicated by T_a, that contains $\gamma_1(a)$. Let C_1 be the hyperbolic circumcircle of T_a. To the sequence $(\gamma_i(a))_{i\in\mathbb{N}}$ there corresponds a sequence of circumcircles $(C_i)_{i\in\mathbb{N}} := (\gamma_i(C_1))_{i\in\mathbb{N}}$. Note that the diameters of these circles, when $\mathbb{B}^2$ is seen in the Euclidean metric, will necessarily tend to zero, and thus also the distances between the C_i's and p will tend to zero, since $\gamma_i(a) \in C_i$. Therefore, there is an $i_0 \geq 1$ such that for all $i \geq i_0$ we have $C_i \subset S(r_0)$ and hence $T_i \subset S(r_0)$.

Step 2. By means of an isometry we may identify $\mathbb{B}^2$ with the upper half-space model $\mathbb{U}^2$ with coordinates $\underline{\zeta} = (z, \zeta) \in \mathbb{R}_{>0} \times \mathbb{R}$. For $x \in S(r_0)$ one then finds by explicit computation $z(x) \leq \text{const.}\, e^{-\rho(o,x)}$. Next we investigate the growth behavior of $H_+ h$ on

$S(r_0)$. For this we use its representation in $\mathbb{U}^2$ which reads [5]

$$\begin{aligned}(H_+h)(z,\zeta) &= \int_{\mathbb{R}} \frac{z^{\Delta_+}}{(z^2+(\zeta-\eta)^2)^{\Delta_+}} h(\eta)d\eta \\ &= z^{-\Delta_++1}\int_{\mathbb{R}} \frac{1}{(1+\eta)^{\Delta_+}} h(z\eta+\zeta)d\eta \\ &\geq \text{ const. } z^{-\Delta_++1}, \end{aligned} \tag{29}$$

because in this case we have $h(z\cdot+\zeta)\geq \text{const.}'>0$ on (α_0,α_1) if $z>0$ is small enough and $\Delta_+>1$. In $\mathbb{U}^2$ the c-neighborhood $N(c,\sigma)$ is simply a cone having σ as symmetry axis. Thus inequality (29) will hold on $S(r_0)$ whenever r_0 is sufficiently large.

Step 3. Now, for $q\in\mathbb{N}_0$ let $j_1=1,\ldots,j_q=q$ and let r_0 be such that inequality (29) is valid. By step 1 we can pick $j_{q+1},j_{q+2},\ldots$ with $j_{q+1}\geq j_q$ so that T_{j_l} approaches $\partial_c\mathbb{H}^2$ in the sector $S(r_0)$. Thus $k_{j_l}\to\infty$ and

$$\mathscr{L}_{X_1}(\lambda k_{j_l})\to\mu(X_1=0)\quad \text{as } l\to\infty,$$

where the r.h.s. is independent of λ. It follows that $\exists n_0(q)\geq q$ such that

$$\sum_{l=1}^{n_0(q)}\left[\log\left(\mathscr{L}_{X_1}(\lambda k_{j_l})\right)+\lambda|T_1|\right]\leq -\varepsilon q,$$

with $\varepsilon>0$ such that $\lambda\leq(-\log(\mu(X_1=0)-\varepsilon)/|T_1|$. Consequently, for $\Lambda_q=\bigcup_{l=1}^{n_0(q)}T_{j_l}\uparrow \mathbb{H}^2$ as $q\to\infty$, we get by Proposition 4.2

$$\tilde{Z}(h,\Lambda_q)/\tilde{Z}(0,\Lambda_q)\leq e^{-\varepsilon q},$$

which proves the assertion choosing choose a subsequence q_n such that $\Lambda_{q_n}\subseteq\Lambda_{q_{n+1}}$. □

Let us finally show that the condition in Theorem 4.1 can always be fulfilled for some $\lambda>0$.

Lemma 4.1 *With the same assumptions as in Theorem 4.1 we have*

$$\mu(X_1=0)<1 \tag{30}$$

Proof. Note that by (23) and $\Delta G(x)\geq 0$, $X_1\geq\,:\exp(\alpha\phi_N):(1_{T_1})$. Thus

$$\mu(X_1=0)\leq\mu(:\exp(\alpha\phi_N):(1_{T_1})=0)<1,$$

since $\mathbb{E}_\mu[:\exp(\alpha\phi_N):(1_{T_1})]=|T_1|>0$. □

A Appendix

There are different isometric models of the d-dimensional hyperbolic space $\mathbb{H}^d$. We give three examples that have been used in this article.

(i) Given the pseudo-Riemannian manifold $(\mathbb{R}^{d+1}, ds_L^2 = dx_1^2 + \cdots + dx_d^2 - dx_{d+1}^2)$, then the Lorentzian model is given by the submanifold

$$\mathbb{L}^d = \{(x_1, \ldots, x_{d+1}) \in \mathbb{R}^{d+1} |\, (x,x)_L := x_1^2 + \cdots + x_d^2 - x_{d+1}^2 = -1,\, x_{d+1} > 0\},$$

equipped with the induced metric. The group $\mathrm{SO}_0(d,1)$ acts transitively on $\mathbb{L}^d$ and the isotropy group of $(0,\ldots,0,1)$ is given by $\mathrm{SO}(d)$ so that this model can also be seen as the homogenous space $\mathbb{L}^d = \mathrm{SO}_0(d,1)/\mathrm{SO}(d)$, a noncompact Riemannian symmetric space.

(ii) The upper half-space model defined by

$$\mathbb{U}^d = \{\underline{\zeta} := (z,\zeta) = (z, \zeta_1, \ldots, \zeta_{d-1}) \in \mathbb{R}^d |\, z > 0\},$$

equipped with the metric $ds_U^2 = (dz^2 + d\zeta_1^2 + \cdots + d\zeta_{d-1}^2)/z^2$.

(iii) The ball model, which is defined through

$$\mathbb{B}^d = \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d |\, \|x\| < 1\},$$

endowed with the metric $ds_B^2 = 4(dx_1^2 + \cdots + dx_d^2)/(1 - \|x\|^2)^2$.

In the ball model every geodesic is either a line through the origin or an arc on a circle which is orthogonal to the sphere $\mathbb{S}^{d-1}$. This sphere with the standard topology provides a natural boundary of $\mathbb{H}^d$, albeit not in the usual sense. To see this, points on $\mathbb{S}^{d-1}$ are identified with appropriate equivalence classes of geodesics. The equivalence class corresponding to $p \in \mathbb{S}^{d-1}$ just comprises all geodesics whose corresponding circles intersect at p. An intrinsic definition can be given by saying that two geodesics $\gamma_1(t), \gamma_2(t), t \geq 0$, are equivalent if $\sup_{t\geq 0} \rho(\gamma_1(t), \gamma_2(t)) < \infty$, cf. [3, Proposition A.5.6]. Therefore one finds a natural boundary (at infinity) given by $\partial_c \mathbb{B}^d = \mathbb{S}^{d-1}$. Obviously the boundary has to be the same for all models. In fact, the following results hold true: $\partial_c \mathbb{U}^d = \{\underline{\zeta} \in \mathbb{R}^d |\, z = 0\} \cup \infty \simeq \mathbb{S}^{d-1}$ and $\partial_c \mathbb{L}^d = (C_L \backslash \{0\})/\sim\, \simeq \mathbb{S}^{d-1}$, where $C_L := \{x \in \mathbb{R}^{d+1} |\, (x,x)_L = 0\}$ and the equivalence relation $\sim$ is given by $x \sim y :\Leftrightarrow x = \lambda y, \lambda \neq 0$.

Hyperbolic spaces are of the form $X = G/K$, where G is a noncompact semisimple Lie group and K is a maximal compact subgroup. By means of the group structure a convolution can be defined

$$f * g(u \cdot o) = \int_G f(v \cdot o) g(v^{-1} u \cdot o) dv, \quad \text{with } o = eK, \tag{31}$$

where dv denotes the left-invariant Haar measure on G. Alternatively, expression (31) can be written in terms of the volume measure $d\overline{v}$ on X. Writing $\overline{u} \equiv uK$, it reads

$$f * g(\overline{u}) = \int_X f(\overline{v}) g(v^{-1} \cdot \overline{u}) d\overline{v}, \tag{32}$$

where v is any representative of $\overline{v}$. In the text above we write $dx \equiv d\overline{v}$. Formula (32) is a consequence of the disintegration formula (9) in [13, Ch.I,§1, Theorem 1.9]. The convolution product belongs to $L^p(X, d\overline{v})$, whenever $f \in L^1(X, d\overline{v}), g \in L^p(X, d\overline{v})$ with $p \in [1, \infty]$, and obeys by Young's inequality

$$\|f * g\|_p \leq \|f\|_1 \cdot \|g\|_p. \tag{33}$$

In particular, the operator $T_f(g) := f * g$ defined on $L^p(X, d\overline{v})$ has norm $\|T_f\| \leq \|f\|_1$. Suppose now that $\mathscr{T}_\varepsilon(f) = \chi_\varepsilon * f$ as in the Remark above, then $\|\mathscr{T}_\varepsilon\| \leq 1$. But any of the L^p's is densely and continuously embedded into the spaces H_+, H_N and therefore $\mathscr{T}_\varepsilon$ has a continuous norm preserving extension to the latter. Due to the isomorphisms $L^2(\mathscr{D}', \mu_\natural) \simeq \bigoplus_{n=0}^\infty \mathcal{H}_n^\natural$, with $\mathcal{H}_n^\natural = H_\natural^{\otimes n}$, see [18, Theorem I.11], there is a natural second quantization $\widehat{\mathscr{T}_\varepsilon}$ of $\mathscr{T}_\varepsilon$ that again satisfies $\|\widehat{\mathscr{T}_\varepsilon}\| \leq 1$ and $\hat{\mathscr{T}}_\varepsilon \to \mathrm{id}$, strongly as $\varepsilon \to 0$.
Finally, we should mention that the Green's function G_+ is given, in the upper half-space model, by

$$G_+(\underline{\zeta}, \underline{\zeta}') = \gamma_+ (2u)^{-\Delta_+} {}_2F_1(\Delta_+, \Delta_+ + \tfrac{2-d}{2}; 2\Delta_+ + 2 - d; -2u^{-1}), \tag{34}$$

where $u = \frac{(z-z')^2 + (\zeta - \zeta')^2}{2zz'}$ and $\Delta_+ = \frac{d-1}{2} + \frac{1}{2}\sqrt{(d-1)^2 + 4m^2}, \gamma_+ = \frac{\Gamma(\Delta_+)}{2\pi^{(d-1)/2}\Gamma(\Delta_+ + 1 - \frac{d-1}{2})}$.
On the other hand, the geodesic distance ρ in the upper half-space model is given by $\cosh(\rho(\underline{\zeta}, \underline{\zeta}')) = 1 + \frac{\|\underline{\zeta} - \underline{\zeta}'\|^2}{2zz'} = 1 + u$, so that (34) becomes

$$G_+(\rho(\underline{\zeta}, \underline{\zeta}')) = \gamma_+ 2^{-2\Delta_+} (\sinh \tfrac{\rho}{2})^{-2\Delta_+} {}_2F_1(\Delta_+, \Delta_+ + \tfrac{2-d}{2}; 2\Delta_+ + 2 - d; -\sinh^{-2} \tfrac{\rho}{2}). \tag{35}$$

From (35) it can be seen that $G_+(\rho) \sim \mathrm{const.}\, e^{-\Delta_+ \rho}$ as $\rho \to \infty$. An alternative expression for (35) is

$$G_+(\rho) = \gamma_+ 2^{-\Delta_+} w^{-\Delta_+} {}_2F_1(\Delta_+, \Delta_+; 2\Delta_+; w^{-1}) = \frac{1}{2\pi} Q_{\Delta_+ - 1}(\cosh \rho), \tag{36}$$

where $w = (1 + \cosh(\rho))/2$ and Q_ν denotes the Legendre function. Equality of expressions (35) and (36) can be seen upon applying the transformation

$${}_2F_1(\alpha, \beta; 2\beta; z) = \left(1 - \frac{z}{2}\right)^{-\alpha} {}_2F_1\left(\frac{\alpha}{2}, \frac{\alpha+1}{2}; \beta + \frac{1}{2}; \left(\frac{z}{z-2}\right)^2\right),$$

to the latter, cf. [6, p.66]. Therefore, the logarithmic singularity of the Green's function is a consequence of

$$Q_{\Delta_+ - 1}(\cosh \rho) \sim -\frac{1}{2} \log(\cosh \rho - 1) \quad \text{as } \rho \to 0,$$

see [6, p.163].

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