Bergische Universität Wuppertal

Fachbereich Mathematik und Naturwissenschaften

Institute of Mathematical Modelling, Analysis
and Computational Mathematics (IMACM)

Preprint BUW-IMACM 12/16

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August 2012

http://www.math.uni-wuppertal.de

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Abstract

Given a periodic solution of a system of ordinary differential equations, the stability of the solution with respect to perturbations in its initial values represents an important property. Appropriate strategies for analysing the local stability of a periodic solution already exist. However, the investigation of a sort of global stability is a more involved task. We introduce a stochastic modelling to analyse a global stability condition, where initial values are substituted by random variables. Now the expected values and the variances of corresponding random processes characterise the stability. We prove sufficient and necessary criteria for the specific global stability based on the stochastic model. Both forced oscillators and autonomous oscillators are examined. Finally, we analyse the global stability for two illustrative examples using this approach.

1 Introduction

We consider systems of ordinary differential equations (ODEs), which exhibit periodic solutions. The local stability of a particular periodic solution with respect to perturbations in its initial values is characterised by well-known concepts. Floquet theory yields sufficient and necessary criteria to determine the asymptotic stability both for autonomous and non-autonomous systems of ODEs, see [3, 11]. Nevertheless, the analysis of a global stability represents a more sophisticated problem.

In this article, a specific concept of global stability is investigated for forced oscillators and autonomous oscillators, respectively. We arrange a stochastic model to analyse this global stability, i.e., to determine if the condition holds. For this purpose, initial values of the solutions are replaced by random variables corresponding to arbitrary random distributions. Thus the solution of the system of ODEs becomes a random process. The global stability of the deterministic solutions implies a specific behaviour of the expected values and the variances of the random solutions in the limit and vice versa. We prove the corresponding implications both for forced systems and autonomous systems.

Oscillators including random effects have already been examined in previous works. The stochastic response of systems of ODEs with random time-invariant parameters is considered in [1, 2, 8, 9, 10]. Systems of ODEs including a time-dependent external forcing term given by a random process are investigated in [6, 7, 12]. Alternatively, noise can be added to a periodic dynamical system in form of a Wiener process, which results in an Itø differential equation, see [5], for example. However, the case of random perturbations in initial values has been considered rather seldom until now.

The article is organised as follows. We define the local and global stability concepts in Sect. 2. We introduce the stochastic model in Sect. 3. The relations between the original deterministic systems and the random-dependent systems are shown. Finally, we discuss two examples in Sect. 4, namely the forced Duffing oscillator and the autonomous Van-der-Pol oscillator.

2 Concepts for Stability

We consider a class of initial values problems of ODEs

$$y'(t;s) = f(t, y(t;s)), \qquad y(0;s) = s \tag{1}$$

with unknowns $y : \mathbb{R} \to \mathbb{R}^n$ and right-hand side $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. Let f be continuous with respect to t and continuously differentiable with respect to y. The initial values $s \in \mathbb{R}^n$ may be restricted to some subset $V \subseteq \mathbb{R}^n$. We assume that the solutions of (1) with initial values $s \in V$ exist for all $t \geq 0$. We call t the time variable, although the meaning of t may be different in some applications. We assume the existence of a periodic solution $\hat{y}$, i.e., it holds

$$\hat{y}(t+T) = \hat{y}(t) \qquad \text{for all } \ t$$

with some period $T > 0$. In addition, let $f(\cdot, y)$ be also periodic for all $y \in \mathbb{R}^n$.

In the following, we apply an arbitrary vector norm on $\mathbb{R}^n$. We assume that f is globally Lipschitz-continuous, which implies the bound

$$\|y(t; s_1) - y(t; s_2)\| \leq \mathrm{e}^{Lt} \cdot \|s_1 - s_2\| \tag{2}$$

for all $s_1, s_2 \in V$ with a Lipschitz constant $L > 0$. Hence existence and uniqueness of solutions of the initial value problems for the ODEs (1) is given. However, several periodic solutions with same period may still exist.

The definition of local stability concepts for periodic solutions of non-autonomous systems of ODEs reads as follows.

Definition 1 *A periodic solution $\hat{y}$ of the system (1) is* Lyapunov-stable*, if and only if*

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall t \geq 0 : \quad \|s - \hat{y}(0)\| < \delta \quad \Rightarrow \quad \|y(t; s) - \hat{y}(t)\| < \varepsilon. \tag{3}$$

Definition 2 *A periodic solution $\hat{y}$ of the system (1) is* asymptotically stable*, if and only if it is Lyapunov-stable and*

$$\exists \theta > 0 : \quad \|s - \hat{y}(0)\| < \theta \quad \Rightarrow \quad \lim_{t \to \infty} \|y(t; s) - \hat{y}(t)\| = 0. \tag{4}$$

Lyapunov-stability and asymptotical stability represent local properties. Sufficient for asymptotical stability (and thus also Lyapunov-stability) is that the monodromy matrix corresponding to the periodic solution exhibits a spectral radius smaller than one.

Now we define an own condition for global stability.

Definition 3 *A periodic solution $\hat{y}$ of the system (1) is said to be* globally stable *with respect to a set $V \subseteq \mathbb{R}^n$, if and only if*

$$\forall s \in V : \quad \lim_{t \to \infty} \|y(t; s) - \hat{y}(t)\| = 0. \tag{5}$$

In this definition, we do not assume $\hat{y}(0) \in V$, since this requirement is not necessary for the conclusions in the next section. Nevertheless, it is reasonable to choose a V containing $\hat{y}(0)$ in practice.

The property (5) has a global meaning, since V may be much larger than a domain of local stability given by (3) or (4). Thereby, the global concept corresponds to an asymptotic stability, i.e., a relatively strong condition. However, the global stability (5) is not sufficient for the Lyapunov-stability (3). Note that different definitions of global stability exist in the literature. We consider the statement (5) in the following.

Likewise, we investigate an autonomous system of ODEs

$$y'(t;s) = f(y(t;s)), \qquad y(0;s) = s \tag{6}$$

including a right-hand side $f : \mathbb{R}^n \to \mathbb{R}^n$ with the analogue properties. Given a periodic solution $\hat{y}(t)$ of this system, the shifted functions $\hat{y}(t+c)$ for $c \in \mathbb{R}$ also represent periodic solutions with the same period. Thus stability is investigated in phase space, where the trajectory of a periodic solution corresponds to a closed curve

$$\Gamma(\hat{y}) := \{\hat{y}(t) : t \in \mathbb{R}\} \subset \mathbb{R}^n. \tag{7}$$

We write Γ instead of $\Gamma(\hat{y})$ for convenience now. For a non-empty set $A \subset \mathbb{R}^n$ and a point $z \in \mathbb{R}^n$, we define the distance function

$$D(z, A) := \inf\{\|z - a\| : a \in A\} \tag{8}$$

using an arbitrary vector norm $\|\cdot\|$. Now the local stability concepts are defined as follows.

Definition 4 *A periodic solution $\hat{y}$ of the system (6) is* orbit-stable*, if and only if the corresponding curve (7) satisfies*

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall t \geq 0 : \quad D(s, \Gamma) < \delta \quad \Rightarrow \quad D(y(t;s), \Gamma) < \varepsilon. \tag{9}$$

Definition 5 *A periodic solution $\hat{y}$ of the system (6) is* asymptotically orbit-stable*, if and only if it is orbit-stable and the corresponding curve (7) exhibits*

$$\exists \theta > 0 : \quad D(s, \Gamma) < \theta \quad \Rightarrow \quad \lim_{t \to \infty} D(y(t;s), \Gamma) = 0. \tag{10}$$

Sufficient (but not necessary) for asymptotic orbit-stability is that $n-1$ eigenvalues of the monodromy matrix corresponding to the periodic solution have a modulus smaller than one.

To analyse a sort of global stability in the autonomous case, one might think of using the approach of a Poincaré map, see [11]. The construction of a manifold of dimension $n-1$ in phase space, which is orthogonal to the trajectory of a periodic solution, allows for choosing initial values inside this manifold. However, the Poincaré map represents a tool for analysing the local stability of a periodic solution. Thus our global stability property applies initial sets $V \subseteq \mathbb{R}^n$ of (possibly) dimension n again.

Definition 6 *A periodic solution $\hat{y}$ of the system (6) with curve (7) in phase space is called* globally orbit-stable *with respect to a set $V \subseteq \mathbb{R}^n$, if and only if*

$$\forall s \in V : \quad \lim_{t \to \infty} D(y(t;s), \Gamma) = 0. \tag{11}$$

The additional requirement $\Gamma \subset V$ is reasonable to discuss neighbourhoods of the trajectory in phase space. However, this requirement is not necessary for the conclusions in the following.

3 Stochastic Model and Implications

In this section, we introduce the stochastic approach and analyse its relation to the original deterministic systems with respect to global stability properties.

3.1 Stochastic Modelling

We assume initial values $s \in V \subseteq \mathbb{R}^n$. Now let $S : \Omega \to V$ be a multidimensional random variable corresponding to a probability space $(\Omega, \mathcal{A}, P)$ with event space Ω, sigma-algebra $\mathcal{A}$ and probability measure P. Therefore V is considered as measurable with respect to the Borel algebra of $\mathbb{R}^n$. We assume $P(S^{-1}(O)) > 0$ for all open sets $O \subset V$, i.e., all open subsets contribute with respect to probability. Using this modelling, the solution of the initial value problem (1) or (6) becomes a random variable in each time point

$$Y(t) : \Omega \to \mathbb{R}^n, \quad \omega \mapsto y(t; S(\omega)). \tag{12}$$

Since y depends continuously on the initial values due to the assumptions on f, the measurability of the random variables is given. Thus the functions (12) represent a stochastic process in time. Now we want to connect the deterministic stability concepts to statements about expected value and variance of this process, i.e., component-wise

$$\begin{aligned} \mathbb{E}(Y(t)) &= \mathbb{E}(y(t;S)), \\ \mathrm{Var}(Y(t)) &= \mathrm{Var}(y(t;S)). \end{aligned} \tag{13}$$

Vice versa, the behaviour of the moments (13) allows for conclusions on the stability of the deterministic solutions.

3.2 Forced Oscillators

The global stability (5) of a periodic solution of the system (1) yields the property

$$\forall s \in V \; \forall \varepsilon > 0 \; \exists t_{\varepsilon,s} > 0 \; \forall t \geq t_{\varepsilon,s} : \quad \|y(t;s) - \hat{y}(t)\| < \varepsilon.$$

However, we need a stronger condition for proving an implication, namely

$$\forall \varepsilon > 0 \; \exists t_{\varepsilon} > 0 \; \forall t \geq t_{\varepsilon} \; \forall s \in V : \quad \|y(t;s) - \hat{y}(t)\| < \varepsilon \tag{14}$$

or, equivalently,

$$\lim_{t\to\infty} \sup\{\|y(t;s) - \hat{y}(t)\| : s \in V\} = 0. \tag{15}$$

This property does not follow from (5). As an explanation, consider the inequality

$$\|y(t;s+\Delta s) - \hat{y}(t)\| \leq \|y(t;s+\Delta s) - y(t;s)\| + \|y(t;s) + \hat{y}(t)\|.$$

The second term on the right-hand side becomes small in the limit due to (5). However, we cannot control the first term on the right-hand side, since it may increase exponentially in time owing to (2).

To achieve the property (14), we also have to assume that a local stability holds. This requirement is not a drawback, since an investigation of global stability makes sense only if local stability is given. Furthermore, we restrict to compact sets of initial values.

Lemma 1 *Suppose that $V \subset \mathbb{R}^n$ is compact and $\hat{y}$ is globally stable with respect to V. If $\hat{y}$ is Lyapunov-stable, then it follows (14).*

Proof:

Let $\varepsilon > 0$ be given. Choose a constant $\delta > 0$ corresponding to ε from the Lyapunov-stability (3). Given an arbitrary $s \in V$, we obtain from the global stability (5) that

$$\exists t_{\delta,s} > 0 \; \forall t \geq t_{\delta,s} : \quad \|y(t;s) - \hat{y}(t)\| < \tfrac{\delta}{2}.$$

Now choose $\tilde{t}_{\delta,s} = kT \geq t_{\delta,s}$ with some $k \in \mathbb{N}$. From the dependence (2), a constant $\sigma > 0$ exists such that

$$\|\Delta s\| < \sigma : \quad \|y(\tilde{t}_{\delta,s}; s + \Delta s) - y(\tilde{t}_{\delta,s}; s)\| < \tfrac{\delta}{2}.$$

Thus we obtain

$$\|\Delta s\| < \sigma : \quad \|y(\tilde{t}_{\delta,s}; s + \Delta s) - \hat{y}(\tilde{t}_{\delta,s})\| < \delta.$$

The Lyapunov-stability implies due to $\hat{y}(\tilde{t}_{\delta,s}) = \hat{y}(0)$

$$\forall \|\Delta s\| < \sigma \ \forall t \geq \tilde{t}_{\delta,s} : \quad \|y(t; s + \Delta s) - \hat{y}(t)\| < \varepsilon.$$

Hence for each $s \in V$, there exists a neighbourhood U_s such that the this property holds for all $t \geq \tilde{t}_{\delta,s}$. Since V is compact, a covering $U_1, \ldots, U_m$ of V exists corresponding to $s_1, \ldots, s_m \in V$. We may select

$$t_\varepsilon := \max \left\{ \tilde{t}_{\delta,s_j} : j = 1, \ldots, m \right\}$$

and thus (14) is satisfied. $\square$

If an arbitrary subset V is given, a reasonable idea seems to approximate V by compact subsets $V_\mu \subset V$, where $P(S^{-1}(V \backslash V_\mu)) < \mu$ holds. However, the solutions belonging to initial values in $V \backslash V_\mu$ may exhibit exponentially growing differences due to (2). Thus we cannot control the remainder of the set V appropriately.

By Lemma 1, we obtain the following conclusion.

Theorem 1 *Suppose that $V \subset \mathbb{R}^n$ is compact and $\hat{y}$ is globally stable with respect to V as well as Lyapunov-stable. Then the expected value and variance of the random process exists for all times and it holds*

$$\begin{aligned} \lim_{t\to\infty} \mathbb{E}(\|Y(t) - \hat{y}(t)\|) &= 0, \\ \lim_{t\to\infty} \mathrm{Var}(\|Y(t) - \hat{y}(t)\|) &= 0. \end{aligned} \tag{16}$$

Proof:

In case of a compact set $V \subset \mathbb{R}^n$, it follows

$$\sup\{\|y(t; s)\| : s \in V\} < \infty \qquad \text{for each } \ t \geq 0. \tag{17}$$

Hence the expected values and the variances in (13) exist for each t. Likewise, it can be shown that the moments in (16) exist for each t.

Furthermore, it holds due to $S(\omega) \in V$

$$\mathbb{E}(\|Y(t) - \hat{y}(t)\|^p) = \int_\Omega \|y(t, S(\omega)) - \hat{y}(t)\|^p \,\mathrm{d}P(\omega) \le \sup_{s \in V} \|y(t, s) - \hat{y}(t)\|^p$$

for each $p \ge 1$. Since the requirements for Lemma 1 are assumed, the condition (15) yields

$$\lim_{t\to\infty} \mathbb{E}(\|Y(t) - \hat{y}(t)\|^p) = 0$$

for each $p \ge 1$. Now we obtain the limits (16) using this equation for $p = 1$ and $p = 2$. □

The conditions (16) provide corresponding limits of the moments (13) of the random process itself by the following lemma.

Lemma 2 *Let $Y : [0,\infty) \times \Omega \to \mathbb{R}^n$ be a stochastic process and $\hat{y} : [0,\infty) \to \mathbb{R}^n$ be a deterministic function. If $\mathbb{E}(\|Y(t)\|^2) < \infty$ holds for all $t \ge 0$, then the conditions (16) are equivalent to the properties*

$$\begin{aligned} \lim_{t\to\infty} \mathbb{E}(Y_j(t)) - \hat{y}_j(t) &= 0, \\ \lim_{t\to\infty} \mathrm{Var}(Y_j(t)) &= 0 \end{aligned} \tag{18}$$

for all components $j = 1, \ldots, n$ of the random process.

We omit the proof of Lemma 2, since it applies simple calculations only.

Thus Theorem 1 also yields the conditions (18), which tell us that the expected value of the stochastic process converges to the globally stable periodic solution and the variance converges to zero.

Now we proceed vice versa by assuming (16). Hence we have to presume the existence of the expected values and variances in case of non-compact sets V. Note that the property (16) is independent of the choice of the vector norm, since all norms are equivalent.

Theorem 2 *Suppose that $\hat{y}$ is a Lyapunov-stable periodic solution of (1). For given $V \subseteq \mathbb{R}^n$, the properties (16) imply the global stability (5) for almost all $s \in V$ with respect to the used probability distribution.*

Proof:

Without loss of generality, we consider a single component $j \in \{1, \ldots, n\}$ of y and Y, respectively, where we omit the index j in the following.

To show the statement, we apply the Chebyshev-inequality in the form

$$P\left[|Y(t) - \mathbb{E}(Y(t))| < c\right] \geq 1 - \frac{\mathrm{Var}(Y(t))}{c^2} \tag{19}$$

for each $c > 0$ and $t \geq 0$. Let $\delta > 0$ and $\eta > 0$ be given. The triangle inequality yields for each $\omega \in \Omega$

$$|y(t; S(\omega)) - \hat{y}(t)| \leq |y(t; S(\omega)) - \mathbb{E}(y(t; S))| + |\mathbb{E}(y(t; S)) - \hat{y}(t)|.$$

Thus we conclude

$$P\left[|Y(t) - \hat{y}(t)| < \delta\right] \geq P\left[|Y(t) - \mathbb{E}(Y(t))| + |\mathbb{E}(Y(t)) - \hat{y}(t)| < \delta\right].$$

The property (16) for the expected value yields

$$\exists t_\delta \geq 0\ \forall t \geq t_\delta: \quad |\mathbb{E}(Y(t)) - \hat{y}(t)| \leq \mathbb{E}(|Y(t) - \hat{y}(t)|) < \tfrac{\delta}{2}.$$

Consequently, we obtain

$$\forall t \geq t_\delta: \quad P\left[|Y(t) - \hat{y}(t)| < \delta\right] \geq P\left[|Y(t) - \mathbb{E}(Y(t))| < \tfrac{\delta}{2}\right].$$

Now the Chebyshev-inequality (19) implies using $c = \frac{\delta}{2}$

$$\forall t \geq t_\delta: \quad P\left[|Y(t) - \hat{y}(t)| < \delta\right] \geq 1 - 4\frac{\mathrm{Var}(Y(t))}{\delta^2}.$$

We use the property (16) or, more precisely, (18) with respect to the variances to obtain a $t_{\delta,\eta} \geq t_\delta$ such that

$$\forall t \geq t_{\delta,\eta}: \quad \mathrm{Var}(Y(t)) < \tfrac{\delta^2 \eta}{4}.$$

It follows

$$\forall t \geq t_{\delta,\eta}: \quad P\left[|Y(t) - \hat{y}(t)| < \delta\right] \geq 1 - \eta. \tag{20}$$

Now we choose $\delta > 0$ corresponding to an $\varepsilon > 0$ from the Lyapunov-stability (3), where the maximum norm is considered without loss of generality. Using some $\tilde{t}_{\delta,\eta} = kT \geq t_{\delta,\eta}$ with an integer $k \in \mathbb{N}$, we achieve

$$P\left[\forall t \geq \tilde{t}_{\delta,\eta}: \ |Y(t) - \hat{y}(t)| < \varepsilon\right] \geq 1 - \eta. \tag{21}$$

This strategy can be done for each component with identical δ and η, i.e., we obtain n possibly different $\tilde{t}_{\delta,\eta}$, where we select the maximum value. Since η can be chosen arbitrarily small, it follows

$$P\left(\{\omega \in \Omega: \ \forall \varepsilon > 0\ \exists t_\varepsilon > 0\ \forall t > t_\varepsilon: \|y(t; S(\omega)) - \hat{y}(t)\| < \varepsilon\}\right) = 1.$$

Since $\varepsilon > 0$ is arbitrary, the global stability is shown for almost all $s \in V$. $\square$

It is interesting to see why the local stability is required again to conclude the result. Without the Lyapunov-stability, we just have the relation (20). Thereby, we only know that a ratio $1-\eta$ of the solutions is inside a δ-neighbourhood beyond some time point. Thus a ratio η may leave the neighbourhood and reenter it at a later time. This allows other solutions to leave the neighbourhood at a later time and reenter. Subsequently, the convergence property (5) may be violated for all $s \in U \subset V$ with some U satisfying $P(S^{-1}(U)) > 0$.

Furthermore, the condition (21) does not imply the strong property (14), since $\tilde{t}_{\delta,\eta}$ may become arbitrarily large for decreasing values η. However, for each $\mu > 0$, a compact set $V_\mu \subset V$ with $P(S^{-1}(V \backslash V_\mu)) < \mu$ exists, where the condition (14) holds for all $s \in V_\mu$.

In contrast to Theorem 1, we have not required a compact set V in Theorem 2. Thus the properties (16) can be seen as strong requirements in case of non-compact sets V. Note again that the existence of expected values and variances has to be presumed in the non-compact case.

3.3 Autonomous Oscillators

Likewise, we perform a corresponding analysis in case of periodic solutions of the autonomous system (6). Thereby, we show a lemma on a stronger property of stability again.

Lemma 3 *Let $V \subset \mathbb{R}^n$ be compact and $\hat{y}$ be a periodic solution of the system (6), which is globally orbit-stable as well as orbit-stable. It follows*

$$\forall \varepsilon > 0 \; \exists t_\varepsilon > 0 \; \forall t \geq t_\varepsilon \; \forall s \in V: \quad D(y(t;s), \Gamma) < \varepsilon \tag{22}$$

for the corresponding trajectory Γ of $\hat{y}$ in phase space.

Proof:

Let $\varepsilon > 0$ be given. We choose a corresponding constant $\delta > 0$ from the criterion of orbit-stability (9). Given an arbitrary $s \in V$, the global orbit-stability (11) yields

$$\exists t_{\delta,s} > 0 \; \forall t \geq t_{\delta,s}: \quad D(y(t;s), \Gamma) < \tfrac{\delta}{2}.$$

In particular, we obtain $D(y(t_{\delta,s};s), \Gamma) < \frac{\delta}{2}$. It follows the existence of a $\hat{t}_{\delta,s} \geq 0$ satisfying

$$\|y(t_{\delta,s};s) - \hat{y}(\hat{t}_{\delta,s})\| < \tfrac{\delta}{2}$$

due to (8). The estimate (2) implies the existence of a constant $\sigma > 0$ depending on δ and s such that

$$\forall \|\Delta s\| < \sigma : \quad \|y(t_{\delta,s}; s + \Delta s) - y(t_{\delta,s}; s)\| < \tfrac{\delta}{2}.$$

We obtain

$$\forall \|\Delta s\| < \sigma : \quad \|y(t_{\delta,s}; s + \Delta s) - \hat{y}(\hat{t}_{\delta,s})\| < \delta$$

and thus

$$\forall \|\Delta s\| < \sigma : \quad D(y(t_{\delta,s}; s + \Delta s), \Gamma) < \delta.$$

The orbit-stability (9) yields

$$\forall \|\Delta s\| < \sigma \; \forall t \geq t_{\delta,s} : \quad D(y(t; s + \Delta s), \Gamma) < \varepsilon.$$

Since V is compact, we obtain again some t_ε independent of s such that

$$\forall s \in V \; \forall t \geq t_\varepsilon : \quad D(y(t; s), \Gamma) < \varepsilon.$$

Thus the formula (22) is shown. □

Lemma 3 implies the following statement for the stochastic model.

Theorem 3 *Let $\hat{y}$ be a periodic solution of (6). If $\hat{y}$ is globally orbit-stable with respect to a compact set V as well as orbit-stable, then it follows*

$$\begin{aligned} \lim_{t\to\infty} \mathbb{E}(D(Y(t), \Gamma)) &= 0, \\ \lim_{t\to\infty} \mathrm{Var}(D(Y(t), \Gamma)) &= 0. \end{aligned} \tag{23}$$

Proof:

The compactness of V implies (17) and thus expected value and variance of the distance function D exist for all $t \geq 0$. The statements (23) follow by the same steps as in the proof of Theorem 1 using $D(Y(t), \Gamma)$ instead of $\|Y(t) - \hat{y}(t)\|$. □

Now we show the conclusion from the stochastic model to the deterministic case. We assume that the expected value and the variance of the distance function exist.

Theorem 4 *Let $\hat{y}$ be an orbit-stable periodic solution of the system (6). If the conditions (23) hold for an arbitrary $V \subseteq \mathbb{R}^n$, then $\hat{y}$ is globally orbit-stable for almost all $s \in V$ with respect to the applied probability distribution.*

Proof:

The Chebyshev-inequality yields the bound

$$P[|D(Y(t),\Gamma) - \mathbb{E}(D(Y(t),\Gamma))| < c] \geq 1 - \frac{\mathrm{Var}(D(Y(t),\Gamma))}{c^2} \tag{24}$$

for each $c > 0$ and $t \geq 0$. The triangle inequality implies

$$D(y(t,S(\omega)),\Gamma) \leq |D(y(t,S(\omega)),\Gamma) - \mathbb{E}(D(y(t,S(\omega)),\Gamma))| + \mathbb{E}(D(y(t,S(\omega)),\Gamma)).$$

for each $\omega \in \Omega$. Let $\delta > 0$ and $\eta > 0$ be given. It follows

$$P[D(Y(t),\Gamma) < \delta] \geq P[|D(Y(t),\Gamma) - \mathbb{E}(D(Y(t),\Gamma))| + \mathbb{E}(D(Y(t),\Gamma)) < \delta].$$

The condition (23) corresponding to the expected value yields

$$\exists t_\delta \geq 0 \; \forall t \geq t_\delta : \quad \mathbb{E}(D(Y(t),\Gamma)) < \tfrac{\delta}{2}.$$

Thus it holds

$$\forall t \geq t_\delta : \; P[D(Y(t),\Gamma) < \delta] \geq P[|D(Y(t),\Gamma) - \mathbb{E}(D(Y(t),\Gamma))| < \tfrac{\delta}{2}].$$

Using the Chebyshev-inequality (24) with $c = \frac{\delta}{2}$ yields

$$\forall t \geq t_\delta : \; P[D(Y(t),\Gamma) < \delta] \geq 1 - 4\frac{\mathrm{Var}(D(Y(t),\Gamma))}{\delta^2}.$$

Now the condition (23) corresponding to the variance gives us a $t_{\delta,\eta} \geq t_\delta$ with

$$\forall t \geq t_{\delta,\eta} : \; \mathrm{Var}(D(Y(t),\Gamma)) < \tfrac{\delta^2\eta}{4}.$$

We obtain

$$\forall t \geq t_{\delta,\eta} : \; P[D(Y(t),\Gamma) < \delta] \geq 1 - \eta.$$

For arbitrary $\varepsilon > 0$, we can choose a corresponding $\delta > 0$ satisfying the property (9) of orbit-stability. It follows

$$P[\forall t \geq t_{\delta,\eta} : \; D(Y(t),\Gamma) < \varepsilon] \geq 1 - \eta.$$

Since η can be chosen arbitrarily small, it follows the global orbit-stability for almost all $s \in V$. □

Thus we retrieve the properties of the non-autonomous case discussed in Sect. 3.2.

4 Illustrative Examples

We investigate two examples: a non-autonomous system and an autonomous system. In the following, the Euclidean vector norm is applied.

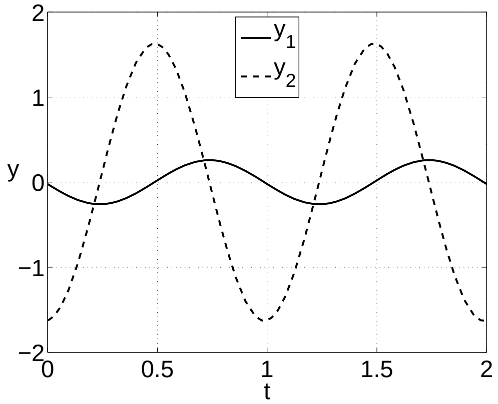

Figure 1: Periodic solution of Duffing oscillator.

4.1 Forced Duffing Oscillator

The Duffing oscillator is often considered as an example in stability analysis and bifurcation theory, see [11]. The mathematical model consists of a scalar ODE of second order. We apply the equivalent system of first order including a forcing term, i.e.,

$$\begin{aligned} y_1'(t) &= y_2(t), \\ y_2'(t) &= -\tfrac{1}{2}y_2(t) - y_1(t) - \kappa y_1(t)^3 + 10\sin(2\pi t) \end{aligned} \tag{25}$$

with a constant $\kappa > 0$.

Firstly, we choose $\kappa = 1$. A locally stable periodic solution $\hat{y}$ with period $T = 1$ exists, which is depicted in Fig. 1. We use the initial values $\hat{y}(0)$ of this solution to define the random initial values $Y_j(0) = \hat{y}_j(0) + S_j$ with two independent identically uniformly distributed random variables $S_1, S_2 \in [-2, 2]$. Approximations for the moments are computed by a Gauss-Legendre quadrature using a 10×10 grid in the random space. An explicit Runge-Kutta method of second order resolves corresponding initial value problems of (25).

Fig. 2 shows the resulting expected values and variances of the first component Y_1. We recognise that the expected value converges to $\hat{y}_1$, whereas the variance converges to zero. The same behaviour appears for $\hat{y}_2$. Hence the results confirm the property (18). The equivalent condition (16) is verified by the moments of $\|Y - \hat{y}\|$ in Fig. 3, where we observe an exponential decay. Thus the results indicate the global stability of the periodic solution with respect to the set

$$V = [\hat{y}_1(0) - 2, \hat{y}_1(0) + 2] \times [\hat{y}_2(0) - 2, \hat{y}_2(0) + 2]. \tag{26}$$

This behaviour can also be seen from the 100 computed solutions, which are

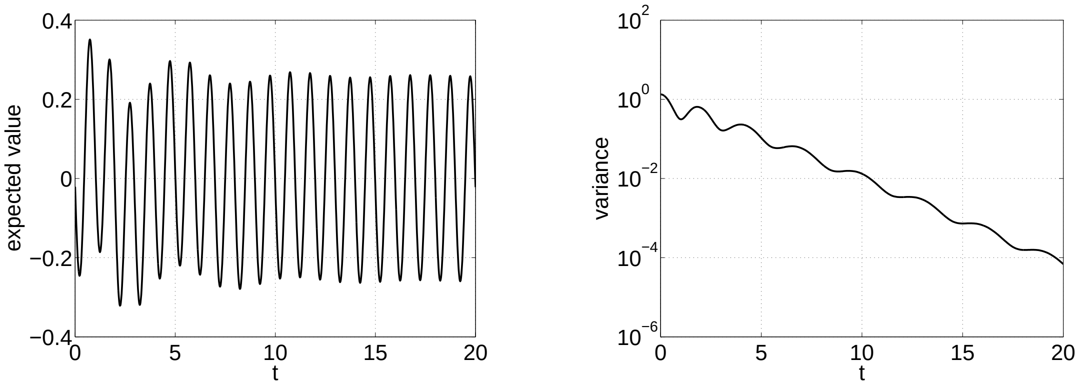

Figure 2: Expected values (left) and variances (right) of Y_1 in Duffing oscillator with random initial values for $\kappa = 1$.

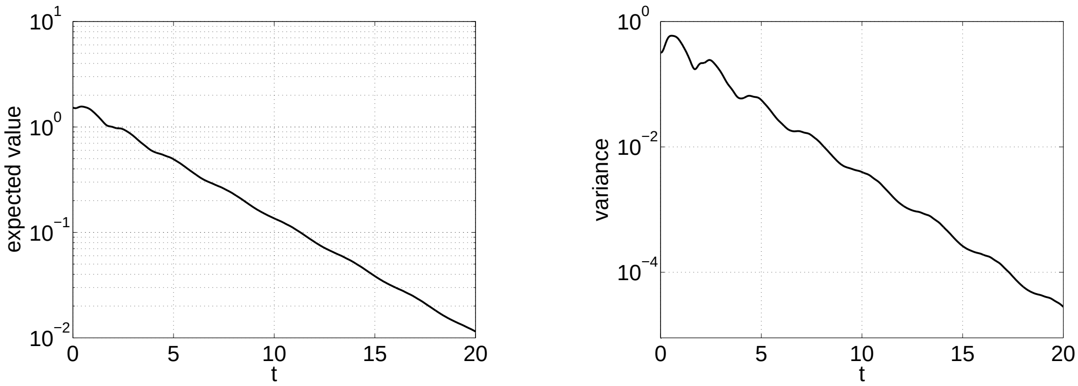

Figure 3: Expected values (left) and variances (right) of $\|Y - \hat{y}\|$ in Duffing oscillator with random initial values for $\kappa = 1$.

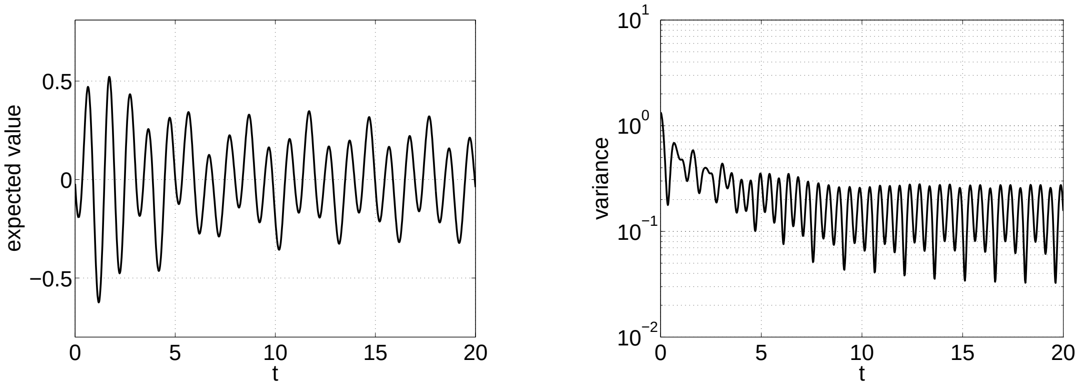

Figure 4: Expected values (left) and variances (right) of Y_1 in Duffing oscillator with random initial values for $\kappa = 10$.

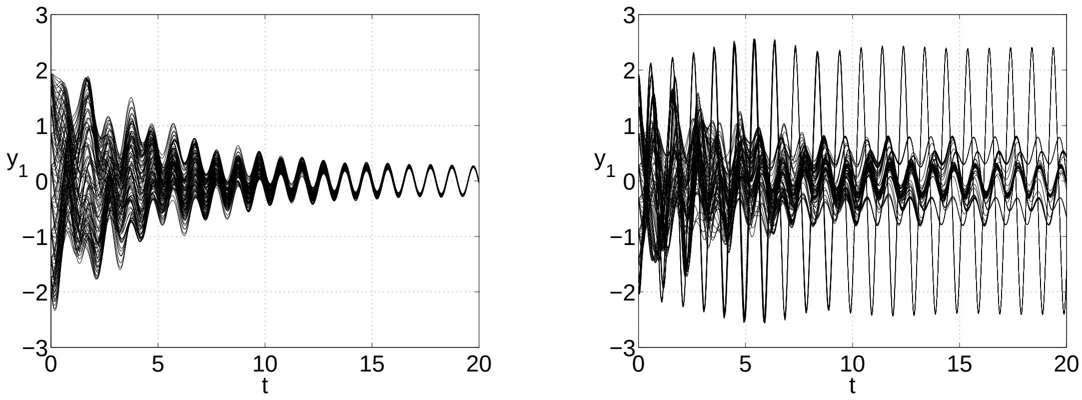

Figure 5: Samples of the solution y_1 for different initial values of the Duffing oscillator in case of $\kappa = 1$ (left) and $\kappa = 10$ (right).

depicted in Fig. 5 (left) for the first component. It follows that all solutions tend to the stable periodic solution.

Secondly, we investigate the case $\kappa = 10$. A locally stable periodic solution $\hat{y}$ exists again, which is nearly the same solution as in the case $\kappa = 1$. Thus we reapply the previous random distribution with slightly different center $\hat{y}(0)$. Fig. 4 illustrates the expected values and variances of the first component Y_1. Now the expected value exhibits a periodic behaviour with a period different from the rate $T = 1$ of the forcing term. Moreover the variance does not converge to zero but becomes periodic. Hence the properties (16) and (18) are not satisfied. We conclude that the investigated periodic solution is not globally stable with respect to the set (26). This behaviour can be understood by observing the computed solutions in Fig. 5 (right) for the first component. Other locally stable periodic solutions exist in the considered domain and thus some of the samples tend to these solutions.

4.2 Van-der-Pol Oscillator

The Van-der-Pol oscillator is also a benchmark in stability theory of ODEs, see [4]. Furthermore, this example has been investigated in case of random parameters in [1, 2]. We consider the particular autonomous system of first order

$$\begin{aligned} y_1'(t) &= y_2(t), \\ y_2'(t) &= -\mu(y_1(t)^2 - 1)y_2(t) - 4\pi^2 y_1(t) \end{aligned} \tag{27}$$

with a constant $\mu > 0$, which determines the stiffness of the problem. A stable periodic solution exists for each μ, which is unique except for a phase shift. The

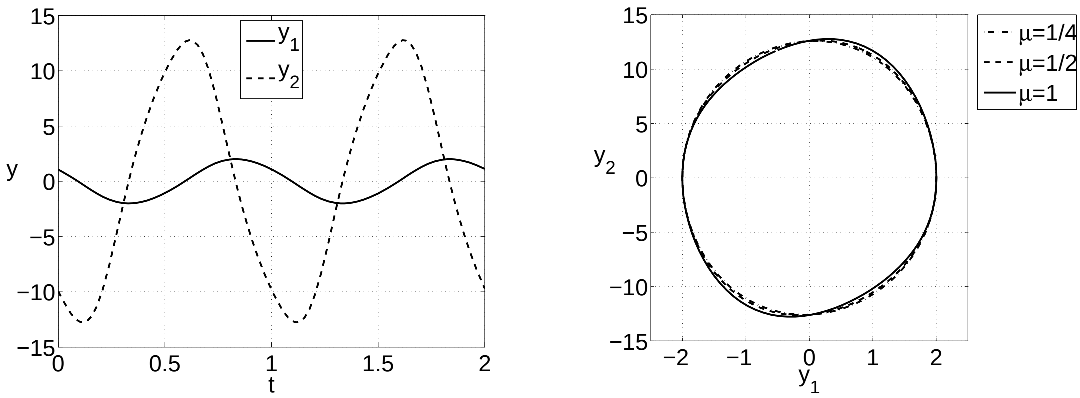

Figure 6: Solution of Van-der-Pol oscillator for $\mu = 1$ in time domain (left) and for different choices of μ in phase space (right).

corresponding period depends on μ. Fig. 6 illustrates the periodic solution for the three different choices $\mu = \frac{1}{4}, \frac{1}{2}, 1$, where the period is approximately $T = 1$.

Motivated by the location of the solutions in phase space, we select the set

$$V := [-4, 4] \times [-20, 20] \tag{28}$$

for a discussion of the global orbit-stability of the periodic solutions. Again we consider a uniform distribution of the initial values within the rectangle (28) using two independent random variables $S_1 \in [-4, 4]$ and $S_2 \in [-20, 20]$. Likewise, the stochastic model is resolved by a Gauss-Legendre quadrature on a grid of size 10×10. A Rosenbrock-Wanner method of second order yields the solutions of initial value problems of (27). Fig. 7 depicts the expected values and the variances for the distance of the random process to each periodic solution in the separate cases $\mu = \frac{1}{4}, \frac{1}{2}, 1$. Since we observe the convergence (23), it follows the global orbit-stability of each solution for almost all initial values in (28). Moreover, the convergence (23) is exponentially fast and becomes faster for increasing μ.

The behaviour of expected values and variances of the distance becomes obvious by observing the trajectories of the solutions for different initial values in phase space. The 100 computed solutions are depicted in Fig. 8 for two different time intervals in case of $\mu = 1$. We recognise that all trajectories tend rapidly to the closed curve, which represents the stable periodic solution. Note that the solution for $y(0) = (0, 0)^\top$ would not converge to this periodic solution but remains constant. However, this particular initial value represents a set of measure zero.

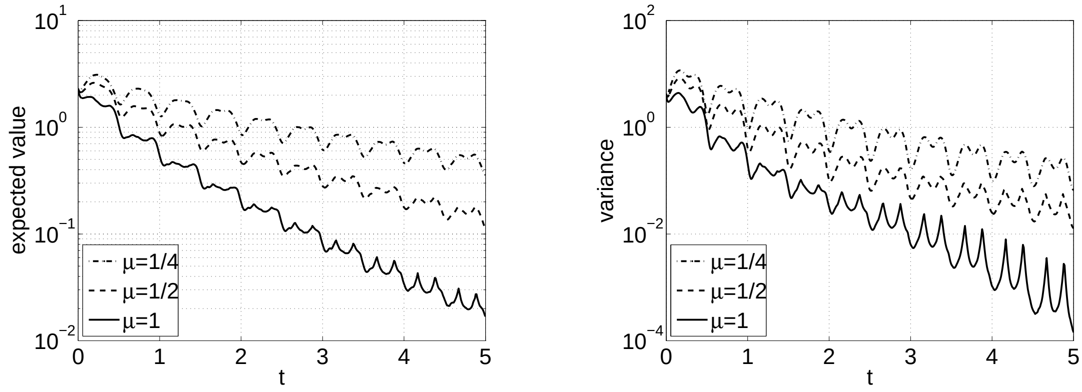

Figure 7: Expected values (left) and variances (right) of $D(Y, \Gamma)$ for Van-der-Pol oscillator with random initial values for different choices of μ.

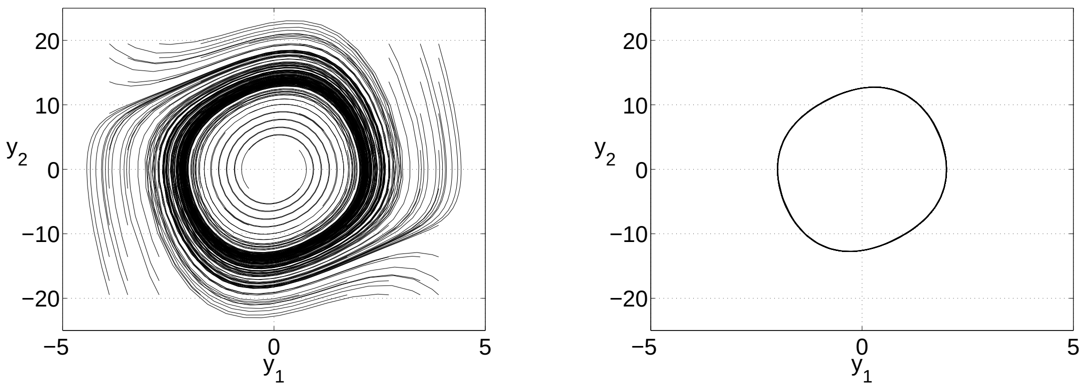

Figure 8: Samples of the solution of the Van-der-Pol oscillator with $\mu = 1$ for different initial values in time interval $t \in [0, 2]$ (left) and $t \in [6, 8]$ (right).

5 Conclusions and Outlook

Conditions for global stability of periodic solutions have been defined for non-autonomous and autonomous systems of ordinary differential equations. We introduced a stochastic model by using random initial values. It follows that the global stability of periodic solutions is related to the limit behaviour of expected values and variances of the corresponding random processes. We showed sufficient and necessary conditions for the global stability. Thereby, a local stability of the periodic solutions is still required. The derived criteria allow for the usage of established methods for the computation of the expected values and variances of random processes to investigate the global stability. However, the global stability cannot be verified without a doubt in the presence of numerical errors of the methods. A generalisation of this modelling to partial differential equations with periodic solutions in time is obvious. Yet a generalisation to periodic solutions of differential algebraic equations is not straightforward, since initial values cannot be chosen arbitrarily but have to satisfy consistency conditions.

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